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CONTINUOUS TRACE MEASUREMENT TECHNIQUE EXPERIMENTAL TEST PHASE II

Final Report

CONTRACT NO. P.O. #70-60218MT

Prepared For
UNIVERSITY OF SOUTH FLORIDA
OFFICE OF SPONSORED RESEARCH
TAMPA, FLORIDA 33620

MAY 1970

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COMPUTER SCIENCES CORPORATION

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JUNE 1970

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ABSTRACT

Performance and accuracy of the Continuous Trace Geometric Model are tested by means of a satellite and station positioning experiment which uses actual observations. The data are right ascension and declination from three passes of PAGEOS observed with BC-4 cameras. The cameras were located at Beltsville, Maryland; Moses Lake, Washington; and Revilla Gigedo Is., Mexico.

The data are preprocessed to eliminate time synchronization and to recover camera orientations and plate continuous traces. Coordinates and relative covariance matrices of the satellite are computed assuming Beltsville and Moses Lake stations to be perfectly located. The position of Revilla Gigedo is evaluated by adjusting its coordinates using a least square procedure applied to the Continuous Trace Technique.

Modifications to and extensions of the analytical development of the Continuous Trace Technique as discussed in the Phase I Report are presented.

The results of the experiment show that the accuracy and practicality of the Continuous Trace Technique is comparable with the accuracy and practicality of other geometrical techniques employing the same data with synchronization. Satellite and station positions are obtained with a precision of a few parts per million which is in agreement with the precision of the measured input data.

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SECTION 1

INTRODUCTION AND SUMMARY

INTRODUCTION

Geodetic tracking cameras are used to observe sunlight reflected from passive satellites to obtain accurate data useful for adjusting the tracking station location. Usual methods of adjustment make use of the geometric aspects of spatial triangulation theory and require highly accurate timing to recover the simultaneous observations. Simultaneity is accomplished through the synchronization of the shutters of the cameras to a common time reference system. As an alternative, the Continuous Trace Technique employs a continuous observation of satellite passes with Earth fixed cameras. Thus, continuous traces are generated on the camera plates. This technique avoids any requirement for synchronization, and the complexity and cost of tracking cameras are thereby reduced.

The objective of the present study is to test the performance and accuracy of the geometric model of the Continuous Trace Technique using actual data.

Two questions are of primary interest for testing purposes. First, is the model capable of producing satellite and station positions from the geometric configurations found in practice? Second, what is the magnitude of the actual error? Error propagation is highly dependent on the geometry of the station/satellite system and on the number of observations. Therefore, the use of real data is preferable to the utilization of simulated observations because (a) no hypothesis is necessary to reproduce the actual satellite trajectory, (b) the data are affected by the actual errors, and (c) a comparison with other methods is not affected by peculiarities in the testing hypotheses.

To the best of our knowledge, no continuous trace observations have been reduced to date. However, a large number of optical observations of passive, sun-reflecting satellites have been made with time synchronized cameras. Among the data available, BC-4 camera observations of PAGEOS, made and reduced by USC&GS, are particularly suitable for testing purposes. The data are expressed as right ascension and declination of satellite position

and an average of 200 points per plate are available. Therefore, a continuous trace representation of the satellite plate image is derivable without introducing bias.

Descriptions of the data employed and of the analysis performed are reported in Section 4. Section 5 describes the implementation of the testing method with computer programs. The final results are presented and discussed in Section 6. The mathematical formulation of the equations employed in the programs and the derivation of the covariance matrix analysis can be found in Section 7. The appendices contain mathematical details of the analytical derivations.

SUMMARY

The methodology employed by Computer Sciences Corporation to perform the test consisted of three steps:

1. Data associated with three events observed from three stations were transformed to Cartesian camera plate coordinates and timing correspondence was eliminated. Thus, we reproduced the original plate measurements of the satellite track without time reference. The orientation of the camera at the epoch was reduced to an Earth fixed system.

2. Plate measurements and camera orientations from two stations were employed to compute the coordinates of some satellite positions on the observed orbital arcs and to estimate the covariance matrices of the coordinate errors.

3. The position of the third station was adjusted by means of a least square procedure to obtain the best fit between plate measurements and computed satellite positions.

Particular care was devoted to the derivation of error covariance matrix estimates to assure the validity of the testing methodology. This derivation was the result of an original study of error propagation theory applied to the Continuous Trace Technique.

The results of the computations show a good agreement between theoretical expectations and practical outputs. Satellite positions were

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obtained with an accuracy comparable to the estimated precision of plate measurements. No degradation was observed. The actual measured data gave the satellite coordinates with a precision of a few parts per million.

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SECTION 2

CONCLUSIONS

Testing of the Continuous Trace Method with real data and comparison of the results with those derived from the same data with established methods has demonstrated the effectiveness of the method for both orbit determination and station location.

The principal limitation occurs with the use of parallel orbital arcs for station adjustment. This is expected from the theory since it corresponds to an insufficient amount of information. Mathematically, parallel arcs create systems of equations which are ill-conditioned and thus give meaningless answers.

The test shows that the Continuous Trace Technique is capable of solving the station adjustment problem when the proper amount of information is available. This conclusion has been reached from single event solutions assuming that only one coordinate is to be corrected. The resulting precision of the correction has an average value of about 20 meters which is in agreement with the precision of the initial data.

Computation of satellite positions shows that the Continuous Trace Technique will be useful for orbit determination. The coordinates of observed orbit points have been recovered with an accuracy of about 60 meters. This error reflects the direct propagation of plate measurements errors since no a priori constraint was available for the orbits.

The tests reported herein demonstrated that the Continuous Trace Technique is a valuable tool for orbit determination and geodetic station adjustment. Use of the technique should significantly reduce the cost of satellite photogrammetry.

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SECTION 3

RECOMMENDATIONS

With the conclusion of this Phase II effort, we have shown that the Continuous Trace Technique (CONTRA) is both feasible and practical for geodetic satellite data processing. The theoretical basis for CONTRA has been analyzed (Phase I) and the geometrical aspects tested experimentally as described in the remainder of this report (Phase II).

The experimental test employed only the geometrical information in the satellite data. This limitation was imposed in order to avoid interference between the problems associated with the determination of orbit parameters and those associated with station positioning. If orbital constraints can be included, the station location accuracy could be improved substantially. Therefore, modification of the technique to include orbital constraints is the logical next step in evaluating the overall accuracy and practicality of CONTRA. A five step approach is recommended:

1. Define an orbit model suitable for use with CONTRA data which can be related to an Earth-fixed coordinate system without requiring a timing reference (see reference 5, pages 16-20).
2. Derive an error model which converts uncertainties in the measured data into uncertainties in the orbital parameters.
3. Develop a statistical method to discriminate those orbit model parameters which can be meaningfully determined.
4. Implement a testing program which provides a definitive understanding of the overall accuracy of CONTRA for station location and orbit determination.
5. Provide NASA with recommendations concerning future utilization of existing camera networks. Determine the realizable benefits associated with maintaining and operating inexpensive camera stations throughout the world.

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SECTION 4

DESCRIPTION OF TESTING METHODOLOGY

In this section we review the fundamental concepts of the Continuous Trace Technique, and we summarize the theory of error propagation as it is applied to the Continuous Trace case. Further, we describe the original input data used to test the Technique.

4.1 REVIEW OF BASIC CONCEPTS

During Phase I of the Continuous Trace Technique analysis, Computer Sciences Corporation investigated the theoretical possibility of deriving geodetic information from continuous trace observations of passive satellites. The theory is based on the following hypotheses:

1. At least three Earth fixed cameras observe overlapping arcs of a satellite pass.
2. The orientation of each camera in an Earth fixed system has been established by independent means to astronomical accuracy.
3. The coordinates of at least two stations are known to geodetic accuracy.

To find the position of the satellite, we employ the observations made from the two known stations. A homological correspondence is established between points on the two associated traces.* Once the homology has been established, the satellite position is obtained by means of an elementary triangulation procedure.

In practice, we represent the trace by tabularizing the trace point coordinates measured from the plate. We proceed as follows (Fig 1):

* The homological correspondence between points on the two traces does not exist if the stations and the observed orbital arc lie in the same plane. In this case, each point on one trace corresponds to a point on the other trace and the Continuous Trace Technique cannot be applied. This difficulty can be avoided by careful observational planning.

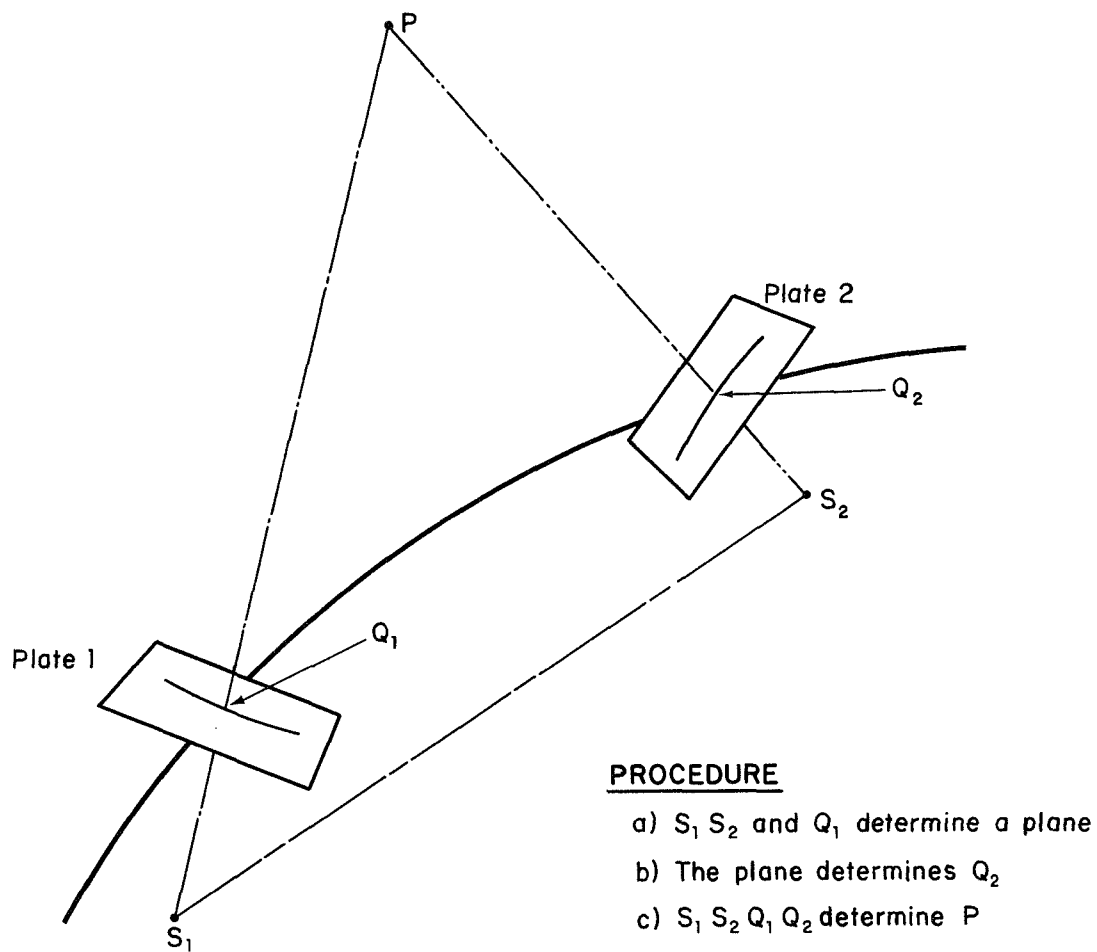


Figure 1 Orbit Points Determination

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1. Point Q_1 is taken from the table corresponding to trace-1. Its coordinates are transformed to the geocentric reference system.
2. The two stations S_1 and S_2 determine a plane through Q_1 .
3. The plane intersects the trace-2 at a point Q_2 . This point is the homolog of Q_1 .
4. The two rays S_1Q_1 and S_2Q_2 then determine the satellite position P.

In theory, the third station coordinates are completely determined if three positions of the satellite are known. In fact, the continuous trace obtained from a station S_3 defines a cone that has its vertex in the third station unknown position. This cone is analytically represented by an equation that relates the coordinates of the station, say X_S, Y_S, Z_S , and the coordinates of the observed satellite, say X, Y, Z .

$$F(X_S, Y_S, Z_S, X, Y, Z) = 0$$

Since we know the coordinates of three satellite positions, we have a system of three equations which can be solved for the unknown X_S, Y_S, Z_S .

In practice, we can obtain more than three satellite positions from the same observed arc. Therefore, we can employ the redundancy of the system of equations to adjust the third station using a least square procedure. However, if the orbital arc is very short we face the problem of an ill-conditioned system because the arc is almost a straight line and the cone becomes indistinguishable from a plane. In this case, only that component of the adjustment normal to the plane can be reliably evaluated. At least three non-parallel arcs are necessary for a complete adjustment of the unknown stations.

4.2 MEASUREMENT ERROR PROPAGATION

There are various errors associated with the measurement of the coordinates from the photographic plate such as those due to shimmer effect, comparator irregularities, and human error. However, we consider only the random residual of these errors. In practice, this residual has an rms

amplitude of a few parts per million.

To determine a satellite position, we need two points - one from each of two plates. There are, of course, four linear components of the error, one for each trace coordinate. However, some of these components do not affect the accuracy of the satellite position determination. The number of error components which must be considered depends on the geometric characteristics of the tracking system as explained in the next paragraph.

First, let us consider the conventional case of simultaneous observations. In this case, we know a priori which pair of images corresponds to the same satellite position. The satellite is at the intersection of two straight lines. The system is over-determined because we have four equations with three unknown coordinates. The redundancy can be employed to reduce the four linear components of error to three. It follows that measurement errors affect all three coordinates of the satellite position and the resulting covariance matrix has rank three. On the other hand, the geometry of the Continuous Trace Technique is very different from the conventional case. The satellite coordinates are evaluated via the procedure delineated in Section 4.1 by determining the homologous points. To determine them, we establish an arbitrary plane through the two stations. Since the placement of the plane is arbitrary, measurement errors normal to the plane are of no consequence. Therefore, only two coordinates of the satellite position are affected by the measurement errors - namely, those which position the satellite within the plane. However, when we use the Continuous Trace Technique we consider the trace as a whole and we disregard the particular identity of any specific point. In other words, if we interchange the positions of two nearby points on the trace, the point coordinates will vary but we cannot recognize any variation in the trace as a whole. Therefore, the error component which interchanges two nearby points does not affect the trace, and we should take into account only that component which is normal to the trace. It follows that only two coordinates of satellite position are affected by measurement errors and that only two of the four measurement error components have some effects. The corresponding covariance matrix will have rank two. The rigorous computation of this covariance matrix is offered in Section 7.3.

4.3 DATA EMPLOYED IN THE TESTING

An experimental test of the performance of the Continuous Trace Technique requires input data coming from real observations of actual satellite orbits. However, to our knowledge, no optical observations in a continuous mode have been reduced to date. Therefore, we decided to employ the results of plate reductions made by USC&GS from observations of PAGEOS passes with BC-4 ballistic cameras located in a worldwide network of stations.

Advantages of employing these data are:

1. Each plate contains an average of 200-400 satellite images spaced every 0.8 sec in time. Therefore, it is feasible to consider these images as belonging to a continuous track of the satellite.
2. Plate measurements have been reduced to right ascension and declination of station-satellite directions with a highly accurate procedure that takes into account astronomic refraction, lens and scale distortions, and plate inclination.
3. The same data have been employed by independent investigators in station positioning problems whose result can be employed for comparison purposes. (Ref 1 & 4)
4. Among the observed events, three events are observed from three stations. This is the minimum of observations necessary to perform the station adjustment.

A disadvantage of using these data is that the plate reduction procedure introduces a statistical correlation among the data of the same plate. However, we remark that: 1) the final solution is still unbiased even if the correlation is neglected, 2) the data we use are plate coordinates which are less correlated than astronomical coordinates, and 3) we employ well separated points of the trace. Therefore, we assume that the correlation among the measured data points is negligible.

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The satellite data employed are the observations of the events 4182, 4236, and 4267 from stations located at Beltsville, Maryland; Moses Lake, Washington; and Revilla Gigedo Is., Mexico.

We assume that we know the positions of Beltsville, Maryland and Moses Lake, Washington exactly. These two stations are used as baseline to evaluate the satellite positions. The third station, Revilla Gigedo Is., Mexico is considered unknown in the testing. Therefore, we do not constrain the adjusted values of its coordinates to any preassumed value. However, we use very good estimates of coordinate starting values in order to avoid the necessity of an iterative procedure in the least square adjustment. Using a good estimate does not harm the test validity because we are mainly interested in the degree to which the covariance matrix of the adjustment reflects the global effect of error propagation.

The station coordinates which were used were obtained from the C-7 Datum (Ref 4) and are given in Table 1.

TABLE 1

	X (m)	Y (m)	Z (m)
Beltsville, Md.	1130773	-4830833	3994706
Moses Lake, Wash.	-2127831	-3785842	4656029
Revilla Gigedo Is., Mex.	-2160983	-5642717	2035347

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SECTION 5

DESCRIPTION OF THE COMPUTER PROGRAMS

The program package employed to test the Continuous Trace Technique reflects the various aspects of the testing methodology described in Section 4.

The operations necessary to perform the test are:

1. Extract the satellite event data from the National Space Science Data Center magnetic tape.
2. Recover plate coordinates of the satellite images and camera orientations.
3. Rotate the plates axes to be parallel to the trace, and evaluate the interpolation polynomials used in computing the derivatives.
4. Find the homological correspondence between the two traces from the known stations for the same event, determine satellite position and evaluate the covariance matrix.
5. Perform the least square adjustment of the third station.

A block diagram that shows the data flow among the programs is given in Fig. 2.

The programs present no particular difficulty from a data processing point of view. Therefore, we present only a brief summary of the operations in the following sections. In Section 7, we describe the mathematical formulas involved.

5.1 DATA EXTRACTION

This program extracts the observational data from the NSSDC Tape and loads a disk file after checking for sequence, format, and completeness. The data stored on the disk are: 1) event and station identifiers, 2) event date, and 3) time, right ascension and declination of each trace point.

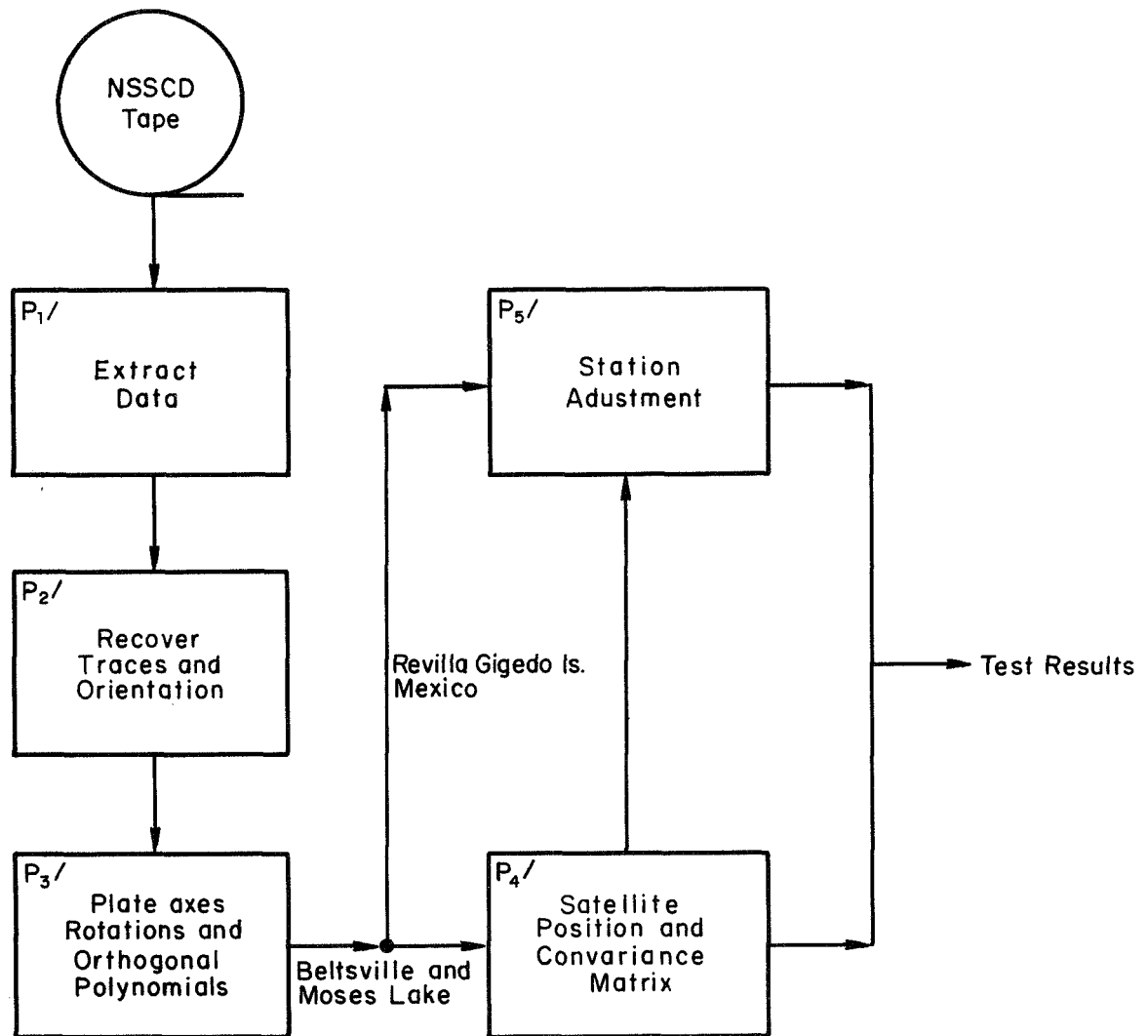


Figure 2 Program Data Flow

5.2 CONTINUOUS TRACE AND ORIENTATION

This program performs the following operations:

1. Corrects the data for phase angle, parallactic refraction and planetary aberration.
2. Transforms right ascension and declination system to Greenwich hour angle and declination system.
3. Evaluates the direction cosines of the rays which project the satellite positions onto the plate.
4. Evaluates the matrix orientation of the established camera system. This system is arbitrarily chosen so that the optical axis is pointing toward the approximate center of the observed arc and is normal to the plate.
5. Calculates the intersections of the projecting rays with the camera plate and computes the image coordinates.

The output data are stored in a disk file. These data for each plate are: 1) Camera Orientation Matrix, and 2) (x, y) coordinates of each trace point.

The time information is employed by this program to convert from the inertial to an Earth fixed system of reference. Time is not employed in subsequent computations. We remark that time inaccuracies are equivalent to an error along the trace and do not effect the trace accuracy as discussed in Section 4.2.

5.3 PLATE AXES ROTATION AND ORTHOGONAL POLYNOMIAL FITTING

The axes of each camera reference system are rotated around the optical axis, z , so that the x axis is parallel to a linear average of the trace. In this way, the straight line component of the continuous trace is reduced to a constant and the possible curvature is easily discriminated.

The coordinates of each trace point are correspondingly transformed. These points are the sampled representation of the continuous trace.

A polynomial is fitted through the points using the method of orthogonal polynomials (Ref 2). This method yields the automatic choice of the best polynomial degree. The fitting is employed to evaluate the

direction cosines of tangents to the trace. These direction cosines are necessary to evaluate the covariance matrix.

5.4 SATELLITE POSITION AND COVARIANCE MATRIX

For each event, ten satellite positions are evaluated. Trace points are randomly taken from the first trace (Beltsville, Maryland) in such a way that the ten points are approximately equispaced.

The homologous point on the second trace (Moses Lake, Washington) is evaluated with respect to each selected point on the first trace. Since the second trace is represented with a large number of sampled points, we employ a simple linear interpolation to solve the homology equation. We avoid using the polynomial fitting for interpolation purposes in order to maintain the statistical independence among the evaluated satellite positions.

The covariance matrix corresponding to each satellite position is evaluated in a rigorous way, taking into account the points discussed in Section 4.2.

5.5 STATION ADJUSTMENT

The coordinates of the third station (Revilla Gigedo Is., Mexico) are adjusted in this program by means of a rigorous least square procedure.

The procedure simultaneously corrects satellite, trace, and third station coordinates by minimizing the weighted squared values of the satellite and trace coordinate corrections. The third station coordinates are considered as unconstrained parameters because: 1) there is no covariance matrix available for the third station, and 2) it would be difficult to isolate the effects of the a priori constraints on the final results of the adjustment.

The equations employed in the adjustment assure that the differences between measured and computed values of the trace coordinates are zero after the adjustment. We do not employ any iteration to solve the equations since the starting values of the third station coordinates are very accurate

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and, therefore, the evaluated corrections are known to be within the linearity limits. In fact, no correction resulted to be larger than 40 m which corresponds to 10 pp million of the station coordinates.

The principal difficulty in the adjustment stems from the ill-conditioned matrix of the station coordinates in their normal system. The traces are almost parallel to each other. An approximate method for checking the matrix condition is employed. For a severely ill-conditioned matrix, only the station coordinates that appear to be meaningful are adjusted. Although we do not claim to have adjusted the station coordinates to their best values, the results are valid for testing purposes.

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SECTION 6

DISCUSSION OF THE RESULTS

Processing real camera data provided significant results for the following three areas:

1. Values of the coordinates for satellite positions and the corresponding covariance matrices.
2. Corrections to the third station coordinates and the corresponding covariance matrix.
3. Estimates of the standard deviations and the degree of the polynomials obtained from the trace fitting procedure. These estimates are employed to evaluate the observational data.

6.1 TRACE CHARACTERISTICS

Estimates of the standard deviation, σ , and the degree of the fitting polynomial, Pd, associated with each event and station are given in Table 2.

TABLE 2

Event	4182		4235		4267	
	$\sigma 10^6$	Pd	$\sigma 10^6$	Pd	$\sigma 10^6$	Pd
Station						
6002	9.32	2	5.81	3	10.46	2
6003	7.12	2	12.86	2	6.44	2
6038	29.3	2	9.40	2	7.85	3

The standard deviation estimates are normalized to a focal length value of 1. The average value of the estimates is

$$\bar{\sigma} = 8.23 \cdot 10^{-6}$$

this is equivalent to a plate measurement error of

$$\sigma_P = 3.7 \mu m$$

for a 450 mm focal length camera.

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This value is in close agreement with values found in the BC-4 camera literature.

The best fitting curve is second degree, a parabola which represents the image of a short arc of a Keplerian orbit. The two third degree cases correspond to the longest arcs observed and probably represent earth rotation effects. However, the coefficients of both second and third degree have very small values since the average curvature of the trace is less than 0.1%.

6.2 SATELLITE POSITIONS

The printouts of the program that computes the satellite position coordinates and correlation matrices are shown in Figures 3, 4, and 5 for the three events analyzed.

The data for each computed point is contained on four lines of printout with the event number and point and line indices being listed at the end of each line. In each group of four lines, the data is presented as follows:

- Line 1: Identifiers
- Line 2: Geocentric position coordinates X, Y, Z (m)
- Line 3: Covariance matrix diagonal terms $CV_{11}, CV_{22}, CV_{33}$ (m^2)
- Line 4: Covariance matrix off-diagonal terms $CV_{12}, CV_{13}, CV_{23}$ (m^2)

The most important data are the correlation matrix values. Some of the matrices in the figures are highly correlated having factors as great as 0.9 between the Y and Z coordinates. This is due to the error propagation characteristics of the Continuous Trace Technique discussed in Section 4.2.

The squared estimates of the standard deviations of the coordinates are represented by the diagonal elements of the matrix. Average rms errors in position coordinates are:

Event 4182	$\bar{\sigma} = 51$ m
Event 4236	$\bar{\sigma} = 88$ m
Event 4267	$\bar{\sigma} = 44$ m

These values agree with the results of the error estimates obtained by fitting polynomials to plate measurements. They correspond to an average

						Event No.	Point Index Line Index
2	3	1	62	6		4182	1 1
-.290996228100	07	-.618676547480	07	.731031768930	074182	4182	1 2
.119828543530	04	.271791390100	04	.381512553180	044182	4182	1 3
.116063721500	04	-.169710874900	04	-.314366669870	044182	4182	1 4
2	3	1	75	20		4182	2 1
-.293147951940	07	-.623731834250	07	.726919819330	074182	4182	2 2
.122978875400	04	.282991476670	04	.369714099440	044182	4182	2 3
.121554795170	04	-.170552594410	04	-.315847013920	044182	4182	2 4
2	3	1	88	32		4182	3 1
-.295633840420	07	-.629550150110	07	.722164539500	074182	4182	3 2
.126654577060	04	.295948296750	04	.356127920490	044182	4182	3 3
.127952822210	04	-.171322139190	04	-.317067871470	044182	4182	3 4
2	3	1	101	46		4182	4 1
-.297760309950	07	-.634559386180	07	.718010890030	074182	4182	4 2
.129809739680	04	.307098523310	04	.344291642640	044182	4182	4 3
.133474082710	04	-.171731799420	04	-.317611644270	044182	4182	4 4
2	3	1	114	53		4182	5 1
-.300042407140	07	-.639922954390	07	.713484934080	074182	4182	5 2
.133197381690	04	.319031585820	04	.331486265010	044182	4182	5 3
.139419005330	04	-.171930625790	04	-.317673811420	044182	4182	5 4
2	3	1	127	67		4182	6 1
-.302147562520	07	-.644873034020	07	.709259403830	074182	4182	6 2
.136328765030	04	.330019806990	04	.319640147290	044182	4182	6 3
.144922859290	04	-.171905630860	04	-.317285072810	044182	4182	6 4
2	3	1	140	81		4182	7 1
-.304714599150	07	-.650919413320	07	.704004107600	074182	4182	7 2
.140137458750	04	.343366149880	04	.305077260830	044182	4182	7 3
.151642945960	04	-.171567276920	04	-.316177240690	044182	4182	7 4
2	3	1	153	94		4182	8 1
-.306799410200	07	-.655835860600	07	.699724480930	074182	4182	8 2
.143248055120	04	.354171641840	04	.293398487110	044182	4182	8 3
.157118028830	04	-.171118640020	04	-.314893329550	044182	4182	8 4
2	3	1	166	105		4182	9 1
-.309491326200	07	-.662195501200	07	.694066603150	074182	4182	9 2
.147234400120	04	.368005456240	04	.278215019330	044182	4182	9 3
.164167122570	04	-.170189993270	04	-.312532103510	044182	4182	9 4
2	3	1	179	118		4182	10 1
-.311530910890	07	-.667013910750	07	.689702187060	074182	4182	10 2
.150234484170	04	.378371254740	04	.266732547860	044182	4182	10 3
.169484881300	04	-.169247920510	04	-.310253565980	044182	4182	10 4

Figure 3 Event 4182 Satellite Positions

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						Event No.	Point Index Line Index
2	3	1	654	35		4236	1 1
.528896385980	06	-.928753065250	07	.464504908110	074236	1 2	
.697891343520	03	.200112295510	05	.393145323130	034236	1 3	
.174542643880	04	-.365334320270	03	-.269102923830	044236	1 4	
2	3	1	674	56		4236	2 1
.535667381900	06	-.934668174210	07	.454410115800	074236	2 2	
.711284245340	03	.210428733000	05	.287975933880	034236	2 3	
.177694309490	04	-.326944892220	03	-.232880880660	044236	2 4	
2	3	1	694	76		4236	3 1
.542481542650	06	-.940725159340	07	.443879613140	074236	3 2	
.725164000410	03	.221302035370	05	.194770082070	034236	3 3	
.180730654550	04	-.286168678680	03	-.191425646110	044236	3 4	
2	3	1	714	93		4236	4 1
.548665328500	06	-.946182003440	07	.434128479720	074236	4 2	
.737749725020	03	.231392130100	05	.125049083320	034236	4 3	
.183252283050	04	-.247836743860	03	-.149697209530	044236	4 4	
2	3	1	734	117		4236	5 1
.555600544250	06	-.952156012600	07	.423117484880	074236	5 2	
.751548076650	03	.242770802880	05	.672818886340	024236	5 3	
.185718711540	04	-.204058908610	03	-.987804144110	034236	5 4	
2	3	1	754	131		4236	6 1
.560534896540	06	-.956496030210	07	.414942484640	074236	6 2	
.761714692630	03	.251222489060	05	.398508340030	024236	6 3	
.187378942950	04	-.171364131070	03	-.584254652530	034236	6 4	
2	3	1	774	150		4236	7 1
.566963450980	06	-.962011876520	07	.404287621020	074236	7 2	
.774622250150	03	.262216976970	05	.252130016880	024236	7 3	
.189206484330	04	-.128674607120	03	-.263504196430	024236	7 4	
2	3	1	794	172		4236	8 1
.573466504260	06	-.967579805370	07	.393178164850	074236	8 2	
.787770841540	03	.273567875450	05	.369118718540	024236	8 3	
.190786748320	04	-.841736021510	02	.592888798250	034236	8 4	
2	3	1	814	191		4236	9 1
.578888671540	06	-.972160429080	07	.383711761940	074236	9 2	
.798649927690	03	.283155130990	05	.697861866970	024236	9 3	
.191851673520	04	-.463994938650	02	.114969119420	044236	9 4	
2	3	1	834	212		4236	10 1
.584747426020	06	-.977060491660	07	.373387028610	074236	10 2	
.810303499860	03	.293569721640	05	.130830046120	034236	10 3	
.192726222510	04	-.553243391620	01	.178687699730	044236	10 4	

Figure 4 Event 4236 Satellite Positions

						Event No.	Point Index Line Index
2	3	1	1444	41	4267		1 1
-.806601363530	06	-.785567244580	07	.649223474640	074267		1 2
.614678023710	03	.348716765170	04	.109755614700	044267		1 3
-.349537994100	03	-.367705606450	02	-.187697132100	044267		1 4
2	3	1	1459	57	4267		2 1
-.807983641600	06	-.792575794070	07	.642299597120	074267		2 2
.622461209180	03	.363841023100	04	.103941595390	044267		2 3
-.338643699780	03	-.502056204790	02	-.186380395640	044267		2 4
2	3	1	1474	68	4267		3 1
-.808992522710	06	-.797795850490	07	.637078284580	074267		3 2
.628292330200	03	.375368144550	04	.996148632420	034267		3 3
-.330095962070	03	-.598663560120	02	-.185169335210	044267		3 4
2	3	1	1489	86	4267		4 1
-.810196174670	06	-.804325173590	07	.630424273990	074267		4 2
.635588192980	03	.390139906920	04	.941658496460	034267		4 3
-.318825233740	03	-.715441920110	02	-.183308352050	044267		4 4
2	3	1	1504	100	4267		5 1
-.811354199630	06	-.810774027440	07	.623727254830	074267		5 2
.642797755430	03	.405060037860	04	.887680063670	034267		5 3
-.307085516840	03	-.626050551320	02	-.181093374520	044267		5 4
2	3	1	1519	114	4267		6 1
-.812235209950	06	-.815811474970	07	.618408108040	074267		6 2
.648428109260	03	.416958739500	04	.845487265090	034267		6 3
-.297509940690	03	-.908972324720	02	-.179092494130	044267		6 4
2	3	1	1534	127	4267		7 1
-.813180606410	06	-.821423550120	07	.612356062830	074267		7 2
.654810377750	03	.430672807760	04	.798372132070	034267		7 3
-.286396215920	03	-.998246380010	02	-.176596152990	044267		7 4
2	3	1	1549	141	4267		8 1
-.814057553250	06	-.826439530070	07	.606976011380	074267		8 2
.660326544010	03	.442814572060	04	.757168013100	034267		8 3
-.276269513530	03	-.107292855490	03	-.174120934920	044267		8 4
2	3	1	1564	153	4267		9 1
-.814923129100	06	-.831655101200	07	.601223666630	074267		9 2
.666137301500	03	.455848010550	04	.713965181430	034267		9 3
-.265286696630	03	-.114791153760	03	-.171283269850	044267		9 4
2	3	1	1579	169	4267		10 1
-.815837130480	06	-.837478364480	07	.5947069231100	074267		10 2
.672617247730	03	.470606074670	04	.666170646200	034267		10 3
-.252671950270	03	-.122641908720	03	-.167702511550	044267		10 4

Figure 5 Event 4267 Satellite Positions

of 8 parts per million of the satellite slant range. (Slant range between PAGEOS and the two stations analyzed was approximately 6500 km.)

6.3 STATION ADJUSTMENT

The principal difficulty encountered in adjusting the coordinates of the third station was caused by the ill-conditioned system of equations. This ill-conditioning was caused by: 1) each trace being almost a straight line, thus limiting the observer to adjusting only one coordinate of the station with each event, and 2) the three observed arcs of the satellite being almost parallel.

The results of adjusting the station coordinates with a single trace are presented in Figure 6. The first three lines for each event correspond to the normal equation* $N_S \Delta S = E_S$ where:

Line 1 contains the known terms E_{S1}, E_{S2}, E_{S3}

Line 2 contains the diagonal entries of the normal matrix N_S
 $N_{S11}, N_{S22}, N_{S33}$

Line 3 contains the off-diagonal terms $N_{S12}, N_{S13}, N_{S23}$

The second group of three lines for each event correspond to the inverse solution of the equation above; that is,

$$\Delta S = (N_S)^{-1} E_S = CV_S \cdot E_S$$

where Line 1 contains $\Delta S_1, \Delta S_2, \Delta S_3$, in meters

Line 2 contains $CV_{S11}, CV_{S22}, CV_{S33}$, in squared meters

Line 3 contains $CV_{S12}, CV_{S13}, CV_{S23}$, in squared meters

All data are expressed with reference to the camera system corresponding to the event.

To avoid serious ill-conditioning effects, we solved only for the station coordinate along the normal to the trace. The undetermined values were left

* Refer to Section 7.6 for a complete development of the normal equation.

EVENT 4182 NO OF POINTS 10

0.15685087913D-04	-0.17114868074D-01	0.22538303723D-03	E_{S1}, E_{S2}, E_{S3}
0.12405254510D-08	0.41894067422D-03	0.72629682179D-07	$N_{S11}, N_{S22}, N_{S33}$
-0.30627081124D-06	0.40435798638D-08	-0.55160924985D-05	$N_{S12}, N_{S13}, N_{S23}$

DEGREES OF FREEDOM 9 SIG 0.1101D 01

0.00000000000D 00	-0.40852724805D 02	0.00000000000D 00	$\Delta S_1, \Delta S_2, \Delta S_3$
0.00000000000D 00	0.23869728139D 04	0.00000000000D 00	$CV_{S11}, CV_{S22}, CV_{S33}$
0.00000000000D 00	0.00000000000D 00	0.00000000000D 00	$CV_{S12}, CV_{S13}, CV_{S23}$

EVENT 4236 NO OF POINTS 10

-0.61880113899D-03	-0.24712784506D-01	-0.10543406759D-03
0.19547541187D-06	0.97426069434D-03	0.17199300528D-07
0.11254631736D-05	0.55317858474D-08	0.40750437277D-05

DEGREES OF FREEDOM 8 SIG 0.2709D 01

-0.30397947709D 04	-0.21854117240D 02	0.00000000000D 00
0.51499861356D 07	0.10332918764D 04	0.00000000000D 00
-0.59492492862D 04	0.00000000000D 00	0.00000000000D 00

EVENT 4267 NO OF POINTS 10

-0.30712963126D-03	0.56185960991D-01	0.11712172487D-02
0.31044155580D-06	0.42621018099D-02	0.18507386841D-05
-0.22172583384D-04	-0.46850321863D-06	0.88807385247D-04

DEGREES OF FREEDOM 9 SIG 0.1254D 01

0.00000000000D 00	0.13182688612D 02	0.00000000000D 00
0.00000000000D 00	0.23462602364D 03	0.00000000000D 00
0.00000000000D 00	0.00000000000D 00	0.00000000000D 00

Figure 6 Single Event Station Adjustment Results

CSC

equal to zero. Although the normal matrix for event 4236 appeared to be better conditioned, a solution for the component along the trace obtained a meaningless result.

Since we employed very reliable station coordinates as starting values in the adjustment, we should expect very small values for the resulting corrections. In fact, the corrections in the direction of the normal to the trace range from -40.8 m to 13.1 m with estimated errors ranging from 52.9 m (r.m.s.) to 17.5 m (r.m.s.) respectively. These values are in agreement with the observation precisions (Section 6.1). However, the value of the along trace correction for event 4236 is very large which shows the ill-conditioning effects. In fact, we obtained anomalous results: -3039.8 m for the correction, and 3750 m (r.m.s.) for the estimated error.

The printouts present also the degrees of freedom and estimates of the standard deviation (SIG) of measurements having unit weight. The values of SIG are nearly 1 because we employed the inverse of the covariance matrices as weighting matrices. The higher value of SIG for event 4236 is a further indication that we cannot solve for two coordinates. Solving for only one coordinate rather than two would reduce value of SIG for event 4236 from 2.7 to 2.3.

A completely different pattern is shown by the output of the global adjustment. In this case, the degrading effect of the ill-conditioned normal matrix on the computations could not be avoided, because we did not have a-priori information about the significance of individual coordinates. The results are presented in Figure 7 with the same reading key as for Figure 6. The principal characteristics of this adjustment are the following:

1. The vector of known terms has an amplitude comparable to that defined for the case of a single event adjustment.
2. The normal matrix is highly correlated.
3. The resulting correction to the coordinates and the corresponding estimated errors are enormous.
4. The standard deviation of the unit weight is much less than 1.

These characteristics confirm that we cannot rely upon the global adjustment results for testing purposes.

CSC

GLOBAL SYSTEM

-0.63940762221D-01	0.50976465868D-01	0.13241829054D-01
0.50453933401D-02	0.40972590587D-03	0.20263165756D-03
0.85222919931D-03	-0.94292006178D-03	-0.23668413280D-03

DEGREES OF FREEDOM 27

SIG 0.2397D 0

0.63542253947D 03	0.16865339766D 04	0.49921632835D 04
0.52127229279D 04	0.25723462068D 05	0.25884685142D 06
0.97454770550D 04	0.35639943420D 05	0.75395628318D 05

Figure 7 Global Adjustment Results

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SECTION 7

MATHEMATICAL DERIVATIONS

This section develops the analytical expressions of the equations employed in implementing the programs.

The equations are, in general, a straightforward derivation of spherical astronomy, projective geometry and Continuous Trace theory as derived in Phase I report. In this Phase II report, particular emphasis is given to analysis and evaluation of satellite position covariance matrices (Section 7.5) and of station coordinates' adjustment (Section 7.6 and 7.7), work which was not treated in detail in Phase I.

7.1 TRACE AND CAMERA ORIENTATION RECOVERY

The first step in the analysis of the Continuous Trace Method consists in recovering the traces on the BC-4 camera plates corresponding to each satellite observation and determining the camera orientation in an Earth fixed reference system.

The data available are right ascension, α , declination, δ , and Greenwich mean time, $t(\text{GMT})$, of the satellite observations provided from the plate reduction procedure by USC&GS [Ref. 3].

Correction to the data should be made for parallactic refraction, aberration, travel time and phase. We employed the values obtained from the results of [Ref. 4]. No correction for nutation was considered [Ref. 5].

The Earth fixed reference system is the Geocentric mean equatorial system (X, Y, Z)

where

- X is in the Greenwich meridian and in the mean equatorial plane
- Y is in the mean equatorial plane and eastward
- Z is the mean polar axis

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In this system the direction from the station to the satellite is expressed by means of spherical longitude ψ and latitude φ . Neglecting the effects of precession, nutation and acceleration of the Earth center gives [Ref. 6]:

1) the spherical longitude

$$\psi = \alpha - \text{GSTO} - 1.0027379t \text{ (sec)}$$

where GSTO is the Greenwich mean time at 0^h UT and 1.0027379 converts the UT to a mean sidereal interval,

and 2) the spherical latitude

$$\varphi = \delta$$

Therefore, the unit vector of the direction from station to satellite has components V_1, V_2, V_3 :

$$V_1 = \cos \varphi \cos \psi$$

$$V_2 = \cos \varphi \sin \psi$$

$$V_3 = \sin \varphi$$

in the average terrestrial system.

The actual orientations of the cameras are not contained in the data available to us. Therefore, it is assumed that each camera is generally aimed toward a point near the middle of the observed satellite orbital arc. This point is somehow arbitrarily chosen since its position is immaterial for generating traces.

We indicate with α_0 and δ_0 the right ascension and declination of the camera axis and with ψ_0 and φ_0 the corresponding spherical longitude and latitude. The orientation of the camera system is therefore assumed to be

z the optical axis

y normal to the optical axis in the mean equatorial plane eastward

x right-handed complement axis

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From this definition the unit vectors of the camera axes are:

$$x_1 = \sin \varphi_0 \cos \psi_0$$

$$x_2 = \sin \varphi_0 \sin \psi_0$$

$$x_3 = -\cos \varphi_0$$

$$y_1 = -\sin \psi_0$$

$$y_2 = \cos \psi_0$$

$$y_3 = 0$$

$$z_1 = \cos \varphi_0 \cos \psi_0$$

$$z_2 = \cos \varphi_0 \sin \psi_0$$

$$z_3 = \sin \varphi_0$$

in the spherical longitude and latitude Earth fixed system.

The system of unit vectors generates the orthogonal matrix that transforms the coordinates of a general vector from the camera system to the Earth spherical system.

This transformation can be expressed by the matrix relationship

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where v_1 , v_2 and v_3 are the components in the camera system. In matrix form we have

$$V = T_{CG} v$$

To recover the coordinates of the trace points on the plate, we assume that the camera system has its origin in the optical center of projection and that the plate plane is normal to the z axis and has equations

$$z = f$$

where f is the camera focal length. A trace point is the intercept of the plate plane with a ray from the camera optical center to the satellite. The

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coordinates (x, y, z) of the trace points are easily obtained in the camera system by means of:

- 1) evaluating the components of the unit vector corresponding to a ray direction in the camera system.

$$v = T'_{CG} V$$

- 2) evaluating the trace coordinates

$$x = f \frac{v_1}{v_3}$$

$$y = f \frac{v_2}{v_3}$$

$$z = f$$

A certain number of randomly chosen points have been generated for each trace. The coordinates of these points and the matrices of the camera systems are the starting data for the numerical analysis of the Continuous Trace Method.

7.2 GENERATION OF TRACE RELATED CAMERA AXES

The camera axes defined in Section 7.1 give a coordinate system which is referenced to the photographic plate by means of the axes x and y. These axes have no relation to the trace of the satellite orbit arc because we chose them arbitrarily.

However, the trace is better represented by means of two axes that are related to the global pattern of the trace. The trace is, in general, almost rectilinear and thus we choose one of the axes, ξ , to represent the linearity of the shape and the other, η , to account for the deviations from linearity (Fig. 8).

We obtain two advantages by using the ξ , η plate system: 1) the numerical computations are more easily programmed, and 2) the deviations from linearity are easily identified and recognized in the final least square adjustment where they have a fundamental effect on the ill-condition of the normal matrix of the station correction system.

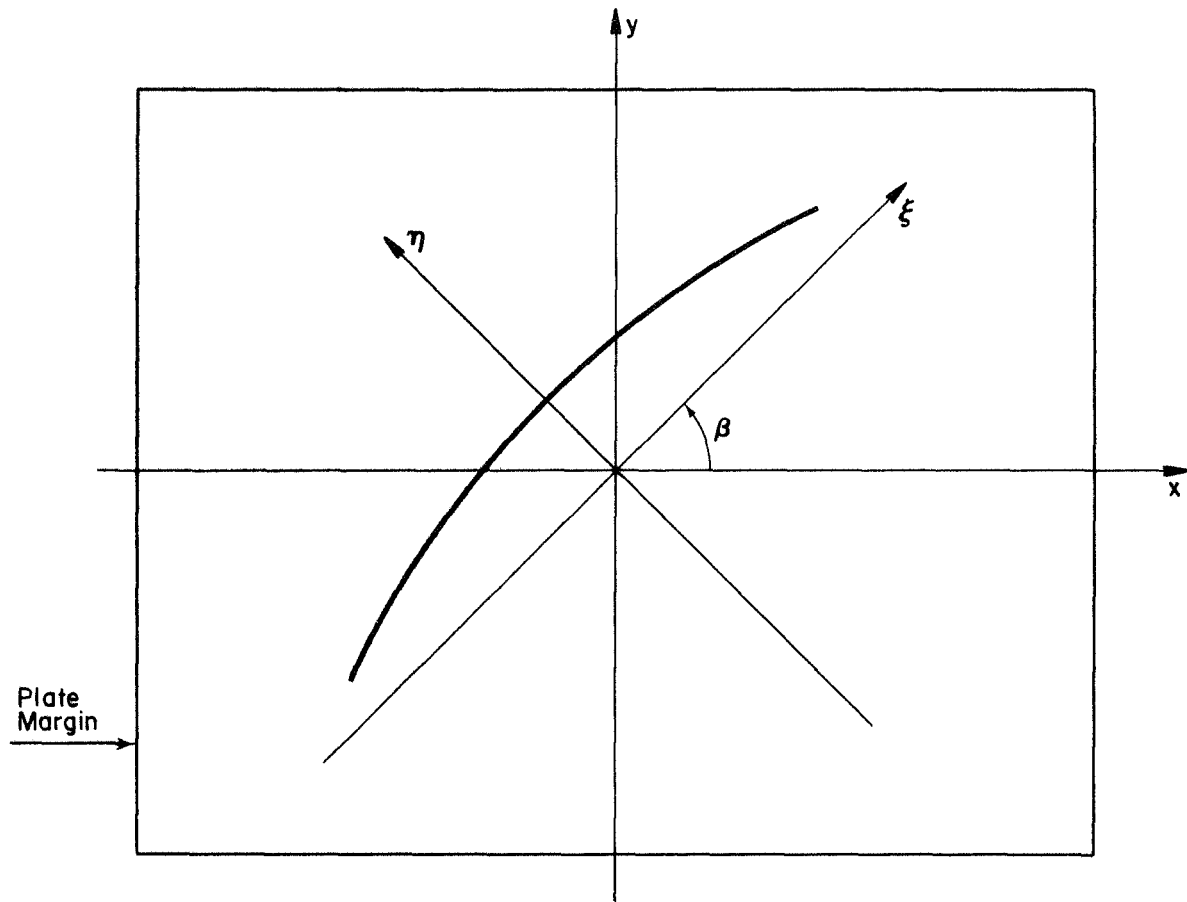


Figure 8 Trace Related Axes

To generate the ξ , η axes, we consider the trace points as mass points. Thereafter, we choose ξ , η to be parallel respectively to the minimum and maximum principal axes of inertia of the mass points. In this way, the coordinates η of the trace contains no component of the linear part. In fact, the coordinate η would have the randomly distributed values of the measurement errors if the true trace were exactly a straight line. The analytical details of the transformation are in Appendix A.

The result of the transformation is a rotation by an angle β around the z axis of the camera and thus it is

$$\nu = \begin{pmatrix} \xi \\ \eta \\ z \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T_B \nu$$

The new matrix T_{RG} that relates the new camera axes to the system is obtained from T_{CG} (previously computed, Section 7.1) and T_B by means of the relations

$$\nu = T_B' \nu$$

$$V = T_{CG} \nu$$

$$V = T_{CG} T_B' \nu = T_{RG} \nu$$

and, thus

$$T_{RG} = T_{CG} T_B'$$

The covariance matrix of the measurements will accordingly transform as

$$C(\xi, \eta) = T_B C(x, y) T_B'$$

7.3 REPRESENTATION OF THE CONTINUOUS TRACE

The traces of the observed satellite orbital arcs consist of a set of points which sample the continuous trace on the photographic plate. However, we need the local tangent of the trace to compute the effects that a variation of the trace itself has on the adjustment of station coordinates. The tangent is evaluated by means of derivatives and this is accomplished by representing the trace with a continuous line.

In general a two dimension continuous line is represented implicitly with an equation relating the coordinates of the points on the line:

$$F(x, y) = 0$$

In the cases at hand, the trace is almost a straight line in the (ξ, η) coordinates (Section 7.2) and, the values of ξ are never greater than the focal length, f , (actually we have $-0.15 f < \xi < +0.15 f$). Therefore, it is possible to neglect every contribution from powers of η higher than one and from products of η by powers of ξ and to represent the trace with η as an explicit function of ξ . Expanding this function in a power series of ξ results in a polynomial representation:

$$\eta = a_0 + a_1 \xi + a_2 \xi^2 + \dots$$

The coefficients of the polynomial are computed by means of a least square fitting through the points of the trace in the (ξ, η) coordinates by means of the method of orthogonal polynomials [Ref. 2]. This method retains only those terms of the series that are significant and minimizes the error resulting from the combination of series truncation and of measurement random noise. Some details are in Appendix B.

The polynomial fittings will furnish us with an estimate of the standard deviation of the measurement. However, the polynomial representation is not employed to evaluate satellite positions and station adjustment in order to maintain the statistical independence of the trace points.

7.4 SATELLITE POSITION EVALUATION

This section deals with the problem of finding the satellite positions corresponding to the traces obtained at two known stations. Satellite positions are necessary to establish the corrections of the third station coordinates, and they are also used in independent investigations such as orbit determination and geopotential analysis. Therefore, the problem is considered as a separate aspect of the Continuous Trace Technique.

We start from the hypothesis that we know:

- 1) the position of two stations S_1 and S_2 which we assume coincident with the optical centers of the corresponding cameras,
- 2) the matrices T_1 and T_2 that relate the camera axes of each station to the Geocentric System, and
- 3) the focal lengths f_1 and f_2 of each camera, and the relationships $\{\eta, \xi\}_i; i = 1, 2$, among the plate plane coordinates of the points of each trace.

Positions of the satellite on the observed orbital arc are obtained in two steps. The first step is the identification of homologous points Q_1 and Q_2 on the two traces. Homologous points are those points on each trace that correspond to the same satellite position. The second step is the evaluation of the coordinates of a satellite position corresponding to two homologous points.

Homologous points are derived from a one-to-one correspondence between the two traces. We consider any plane in the family of planes through S_1 and S_2 , and we assume that the orbital arc intersects this plane at B [Figure 9]. The plane will in general intersect each photographic trace at a point, say Q_1 and Q_2 respectively for S_1 and S_2 . Since the points Q_1 and Q_2 are on the same plane through S_1 , S_2 and B, it follows that Q_1 and Q_2 are the images of B. Therefore, Q_1 and Q_2 are homologous points.

To identify homologous points in practice, we choose any point Q_1 on the first trace and we consider the plane through S_1 , S_2 and Q_1 . The intersection between the plane S_1 , S_2 , Q_1 and the second trace is the homologous Q_2 of Q_1 .

The point Q_2 is obtained by means of the homology equation, that in vectorial notation has the form

$$(\overline{S_1 Q_1} \times \overline{S_1 S_2}) \cdot \overline{S_2 Q_2} = 0$$

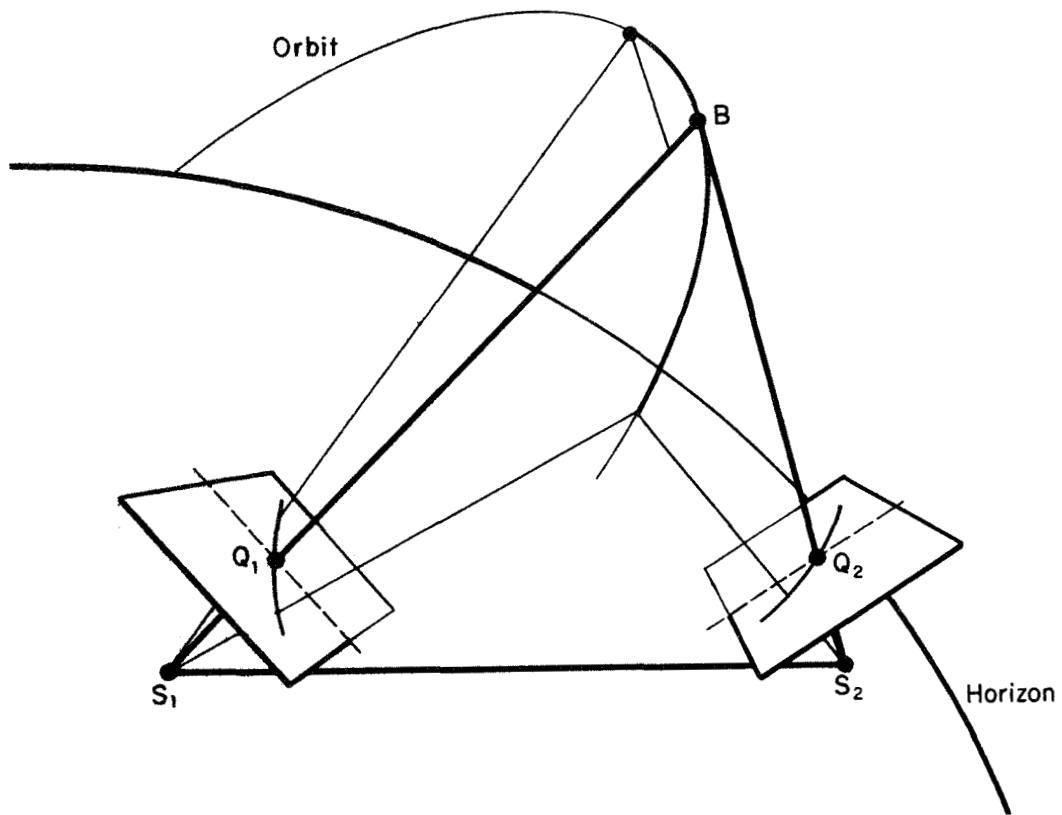


Figure 9 Trace Homology Determination

The position B of the satellite is obtained directly by means of triangulation in the plane through S_1 , S_2 , Q_1 and Q_2 . It is the intersection of the two straight lines through S_1 and Q_1 and through S_2 and Q_2 [Figure 10]. In vectorial notation we have

$$\overline{B} = \overline{S}_1 + C_B \overline{S}_1 Q_1$$

where the coefficient C_B is:

$$C_B = \frac{D}{|\overline{S}_1 Q_1|} \frac{\sin \gamma_2}{\sin \gamma_3}$$

Details for the solution of the homology equation are reported in Appendix C.

The Continuous Trace Technique fails if the homology equation vanishes or is unsolvable. The equation vanishes if all points of the two traces lie on the same plane through S_1 and S_2 . This is the case when the observed orbital arc is entirely contained in a plane through S_1 and S_2 . The equation is unsolvable if no homology exists between the two traces. This occurs if the two stations observe nonoverlapping portions of the same orbit.

7.5 MEASUREMENT ERRORS AND SATELLITE POSITION COVARIANCE MATRIX

The covariance matrix of satellite position is the statistical representation of the random errors in the coordinates of the satellite. It is defined as the statistical average of the product of the random error vector and its transpose. In symbols

$$CV = \langle \partial B \cdot \partial B' \rangle$$

These random errors in the satellite coordinates are due to the errors in measuring the coordinates of the traces on the plates. In order to derive the covariance matrix of satellite position, we assume that the covariance matrix of each plate measurement is known. In order to simplify the computations, we further assume: 1) that the measurement errors have a

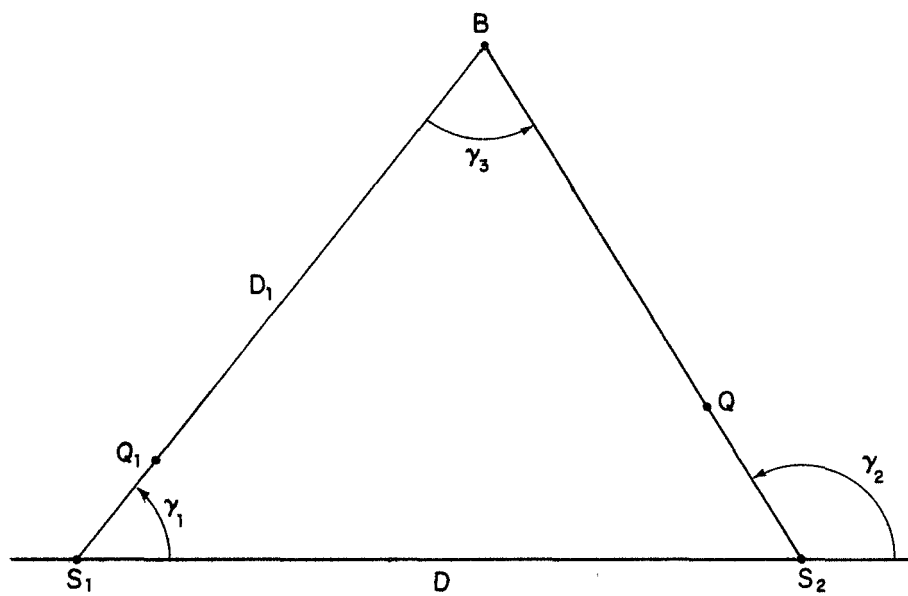


Figure 10 Plane Triangulation

zero average and are statistically independent, and 2) that the standard deviation, σ^2 , is a constant, independent of the direction, for each plate. Therefore, the covariance matrix for every point of a plate is

$$CT = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the camera reference system.

To evaluate CV we should compute the error ∂B as a function of the measurement errors. This is accomplished by differentiating the expression for B obtained in Section 7.4

$$\overline{B} = \overline{S}_1 + C_B \overline{S}_1 Q_1$$

To obtain B we follow four steps:

- 1) Arbitrarily establish a plane through S_1 and S_2 .
- 2) Find the homologous points Q_1 and Q_2 on the two traces.
- 3) Evaluate the lengths of the two vectors $S_1 Q_1$ and $S_2 Q_2$ and the angles γ_1 and γ_2 between these vectors and the vector $S_1 S_2$.
- 4) Compute the coordinates of B.

The variables affected by measurement errors are the lengths and angles evaluated in step 3. These variables affect the position of B in the plane through S_1 and S_2 . Therefore, the error ∂B is entirely in this plane and has no component along the normal to the plane. Actually, the two angles γ_1 and γ_2 are the only independent variables of the triangulation, and ∂B can be completely defined in terms of the errors $\partial \gamma_1$ and $\partial \gamma_2$.

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The analytical details and the relationship between angle errors and measurement errors are described in Appendix D. The resulting covariance matrix of the satellite position can be represented by

$$CV = a^2 (b_1^2 M_1 C_1 M_1' + b_2^2 M_2 C_2 M_2')$$

where a, b_1, b_2 are coefficients

M_1 and M_2 are two matrices

The definition of these quantities is presented in Appendix D.

The matrices C_1 and C_2 represent the covariance matrices of measurements errors in the two plates relative to the continuous traces. The derivation of these matrices deserves a particular discussion.

The coordinates ξ, η of a point on a photographic plate are affected by measurements error $\partial\xi, \partial\eta$ that we assumed to be statistically independent with a constant standard deviation. This is also true if we consider a particular point of the trace. However, we do not use the points of a continuous trace as if they have an individual identity; we use the trace as a whole. In other words, if we let every point of the trace slide along the trace itself, we change the coordinates of each point of the trace but the trace as a whole is completely unaffected. Therefore, we can disregard those measurement errors that do not affect the trace considered as a continuous curve.

To state this observation analytically, let us consider point p on the trace and assume that measurement errors have only first order effects. Construct two axes t and n which are tangent and normal to the trace at point p (see Figure 11). An error in measuring the coordinates of p has components ∂n normal to and ∂t along the trace. The component ∂t does not affect the trace and should be deleted. Thus in the local axes t, n , the covariance matrix for each point of the trace is

$$\begin{pmatrix} 0 \\ \partial\eta \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \partial\eta & 0 \end{pmatrix} = \sigma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

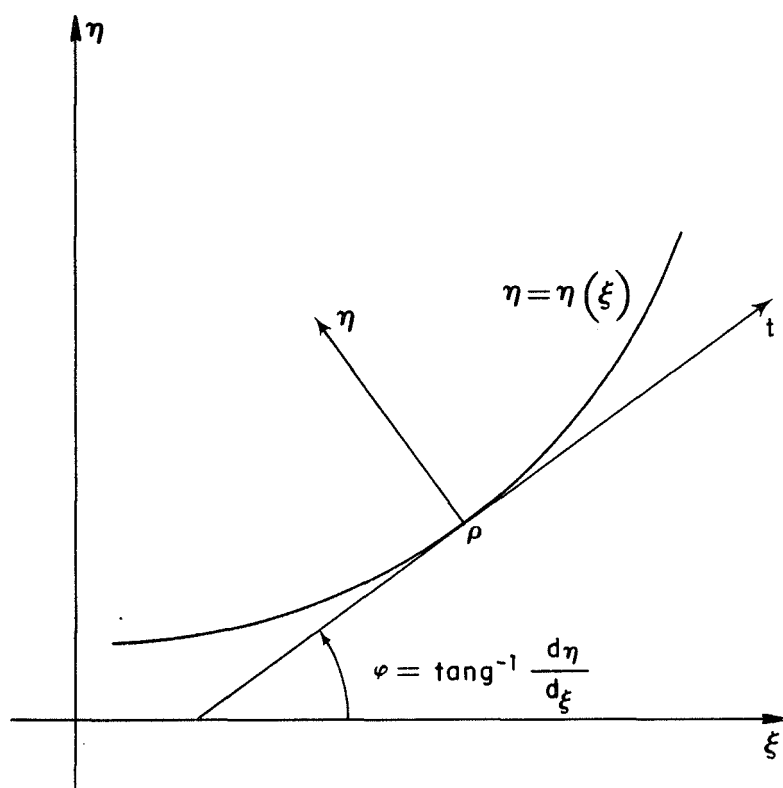


Figure 11 Propagation of Plate Error to the Trace

CSC

Transformation to plate axes ξ, η

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{1 + \eta'^2}} \begin{pmatrix} 1 & -\eta' \\ \eta' & 1 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix}$$

gives the covariance matrix

$$C = \frac{\sigma^2}{1 + \eta'^2} \begin{pmatrix} \eta'^2 & -\eta' \\ -\eta' & 1 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix}$$

where

$$\eta' = \frac{d}{d\xi} \eta(\xi)$$

is the derivative of the function $\eta = \eta(\xi)$ which represents the trace.

In summary, the analysis of the errors in the satellite position shows that:

- 1) Satellite position error has only two components different from zero, and
- 2) One of the components of each plate measurement error has no effect.

7.6 LEAST SQUARE ADJUSTMENT OF STATION COORDINATES

Station coordinates are adjusted by means of a least square fitting between the measured trace relative to the station and the trace computed from the satellite positions. We assume that the orientation of the camera is known along with the coordinates of the trace points in the camera system. Furthermore, we assume that the station coordinates are accurately known a priori.

CSC

Let us consider N positions of the satellite on the observed orbital arc, say P_1, \dots, P_N . In order to simplify the analytical expressions, we will express station and satellite positions in the camera system. In this system, a satellite point P will have coordinates

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

the station will have coordinates

$$S = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}$$

and a point on the plate will have coordinates

$$p = \begin{pmatrix} \xi \\ \eta \\ f \end{pmatrix}$$

The true values of these coordinates should satisfy the constraints that: 1) the points S , p , and P lie on a straight line, and 2) the point p lies on the trace. Therefore, we have the straight line equations

$$\begin{cases} G(S, P, p) = f(y - y_s) - \eta(z - z_s) = 0 \\ \xi = f \frac{(x - x_s)}{(y - y_s)} \end{cases}$$

CSC

and the trace equation

$$\eta = \eta(\xi)$$

The true values of the coordinates are given by the approximate values plus a correction

$$P = P_o + \Delta P$$

$$S = S_o + \Delta S$$

$$p = p_o + \Delta p$$

where index o refers to the approximate values.

By means of least square method, we compute ΔP , ΔS , and Δp such that the variance

$$V^2 = \sum_{i=1}^N \Delta P_i' w_i \Delta P_i + \sum_{i=1}^N \Delta p_i' v_i \Delta p_i + \Delta S' w_S \Delta S$$

is a minimum, with the linearized constraints

$$G(S, P_i, p_i) = G_o + G_P' \Delta P_i + G_S' \Delta S + G_p' \Delta p_i = 0 \quad i = 1, \dots, N$$

We employ Lagrange multipliers, λ_i , to solve the problem of minimization. Thus, we have the system of linear equations

$$\begin{pmatrix} w_i \Delta P_i & + G_P' \lambda_i & = 0 \\ v_i \Delta p_i & + G_p' \lambda_i & = 0 \end{pmatrix} \quad i = 1, \dots, N$$

$$w_S \Delta S + \sum_{i=1}^N G_S' \lambda_i = 0$$

$$(G_P' \Delta P_i + G_p' \Delta p_i + G_S' \Delta S + G_o = 0) \quad i = 1, \dots, N$$

In these equations G_P , G_p , G_S indicate the vectors of partial derivatives of the function $G(S, P, p)$ with respect to the coordinates of P , p , and S respectively. The matrices w_i , v_i , w_S are the weighting matrices of the corrections ΔP , Δp and ΔS respectively.

CSC

Since we are interested in determining station positions, we solve the system for the unknown ΔS . The weighting matrices are the inverses of the covariance matrices of the errors of the corresponding quantities. Therefore,

$$\Delta P_i = - CV_i G_P \lambda_i$$

$$\Delta p_i = - C_i G_p \lambda_i$$

and

$$\lambda_i = \frac{1}{C_{\lambda i}} [G_o + G_S' \Delta S]$$

The coefficient $C_{\lambda i}$ is given by

$$C_{\lambda i} = G_P' CV_i G_P + G_p' C_i G_p$$

Substitution of the expression of λ_i in the equation for ΔS gives

$$\left[w_S + \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_S G_S' \right] \Delta S = - \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_o G_S$$

This is the normal equation of ΔS

$$N_S \Delta S = -E_S$$

where

$$N_S = w_S + \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_S G_S'$$

$$E_S = \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_o G_S$$

If the normal equation can be solved, we have the corrections to the station coordinates. The covariance matrix of the station coordinates is given by

$$CV_S = \sigma_o^2 N_S^{-1}$$

CSC

where σ_o^2 is the standard deviation of the unit weight observation

$$\sigma_o^2 = \frac{v^2}{N-3}$$

and

$$v^2 = \sum_{i=1}^N \frac{G_o^2}{C_{\lambda i}} - \Delta S' N_S \Delta S$$

The number of degrees of freedom of the adjustment is clearly $N-3$.

The results are extended to the observations of n passes if we add, term by term, the equations relative to each pass. However, we should express the adjustment ΔS to the same system of coordinates by means of the camera orientation matrices T_j , $j = 1, \dots, n$. Therefore, we obtain the compound system of equations

$$N_{ST} \Delta S = - E_{ST}$$

$$N_{ST} = \sum_{j=1}^n T_j N_{Sj} T_j'$$

$$E_{ST} = \sum_{j=1}^n T_j E_{Sj}$$

Details of the computations of the derivatives are reported in Appendix E.

The normal equation relative to a single pass is singular if the trace is a straight line. In this case, only one component of the correction ΔS can be evaluated, namely the component along the η axes.

In practice, the trace is almost a straight line and the normal equation can be ill-conditioned. To solve the normal equation, we employ a test that detects the ill-conditioned cases. Thereafter, we solve the equation only for the meaningful correction components.

7.7 CASE OF A STRAIGHT LINE TRACE

We can easily discuss the case where the trace is a straight line because the plate axes are parallel to the trace [Section 7.2].

The trace equation is

$$\eta = \eta_o$$

where η_o is a constant, and the trace derivative $\eta' = 0$.

Let us assume that no reliable information about ΔS is known. Therefore, $w_S = 0$ and the normal matrix is singular. In fact [Appendix E]

$$N_S = \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_S G_S'$$

and

$$G_S G_S' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f^2 & -f\eta_o \\ 0 & -f\eta_o & \eta_o^2 \end{pmatrix}$$

when $\eta' = 0$.

By means of a rotation around the ξ axes we can obtain $\eta_o = 0$. The system thus becomes

$$f^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{i=1}^N \frac{1}{C_{\lambda i}} \Delta S = f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sum_{i=1}^N \frac{G_o}{C_{\lambda i}}$$

It is apparent that we have sufficient information to adjust only one coordinate of ΔS , namely the coordinate along the η axes. The correction is

$$(\Delta S)_\eta = \frac{\frac{1}{f} \sum_{i=1}^N \frac{G_o}{C_{\lambda i}}}{\sum_{i=1}^N \frac{1}{C_{\lambda i}}}$$

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The standard deviation for the unique component is

$$\sigma_{\Delta S \eta}^2 = \frac{\sigma_{o\eta}^2}{N f^2 \sum_{i=1}^N \frac{1}{C_{\lambda i}}}$$

where the standard deviation of the unit weight observation is

$$\sigma_{o\eta}^2 = \frac{1}{N-1} \left(\sum_{i=1}^N \frac{G_o^2}{C_{\lambda i}} - \left(\sum_{i=1}^N \frac{G_o}{C_{\lambda i}} \right)^2 \left(\sum_{i=1}^N \frac{1}{C_{\lambda i}} \right)^{-1} \right)$$

since we have in this case N-1 degrees of freedom.

All the coordinates of ΔS can be adjusted only if we have at least three observed passes which are not parallel to a common plane.

REFERENCES

1. C. R. Schwarz, "The Use of Short Arc Constraints in the Adjustment of Geodetic Satellite Data", OSU Department of Geodetic Science, Report No. 118, January 1969
2. A. Vallone, L. A. Gambino, "Fitting the BC-4 Camera Data Using Orthogonal Polynomials", 7th Western National Meeting, American Geophysical Union, San Francisco, California, December 1968
3. F. D. Hotter, "Preprocessing Optical Satellite Observations", OSU Department of Geodetic Science, Report No. 82, April 1967
4. A. Mancini, et al, "National Geodetic Satellite Program (NGSP) Station Solution", Hawaii Institute of Geophysics and Geonautics, Incorporated, October 1969
5. Computer Sciences Corporation, "Continuous Trace Measurement Technique Analysis, Phase 1", Contract No. P. O. #69-30165, May 1970
6. I. I. Mueller, "Spherical and Practical Astronomy", Frederick Ungar Publishing Company, New York, N. Y., 1969

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APPENDICES

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APPENDIX A

TRANSFORMATION TO PRINCIPAL AXES OF INERTIA

Measurement of a photographic plate containing a continuous trace results in a set of n points P_i , each of which can be expressed in terms of the plate coordinates (x, y)

$$P_i = (x_i, y_i) \quad i = 1, \dots, n$$

By considering each of the observed points to be a unit of mass, we can define a set of principle axes of inertia for the entire set. To apply this data to the continuous trace, we need then to refer the individual points to a system of axes (ξ, η) which are parallel to the principal axes of inertia and which produce a minimum moment of inertia about ξ . This is obtained by a rotation of the x, y system by an angle β :

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since the principal axes of inertia have their origin in the center of mass of the system, we compute the coordinates of the center by means of

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Therafter, the principal axes of inertia are obtained by the equation:

$$\sum_{i=1}^n (\xi_i - \xi_c) (\eta_i - \eta_c) = 0$$

and the moment of inertia around ξ is a minimum if

$$\sum_{i=1}^n (\xi_i - \xi_c)^2 > \sum_{i=1}^n (\eta_i - \eta_c)^2$$

Substituting the values of ξ and η as a function of x and y reduces the equation defining the principal axes to

$$b^2 J_{xy} + b (J_{xx} - J_{yy}) - J_{xy} = 0$$

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where $b = \frac{\sin \beta}{\cos \beta}$

$$J_{uv} = \sum_{i=1}^n (u_i - u_c) (v_i - v_c) \quad u = x, y \quad v = x, y$$

Solving the equation gives

$$\sin \beta = \text{sign} (J_{xy}) \sqrt{\frac{\sqrt{D} - (J_{xx} - J_{yy})}{2\sqrt{D}}}$$

$$\cos \beta = \sqrt{\frac{\sqrt{D} + (J_{xx} - J_{yy})}{2\sqrt{D}}}$$

where $D = (J_{xx} - J_{yy})^2 + 4 J_{xy}^2$

The inequality, which assures a minimum moment of inertia about ξ , is always satisfied since $\sin \beta$ has the sign of J_{xy} .

APPENDIX B

BEST POLYNOMIAL FITTING BY MEANS OF
ORTHOGONAL POLYNOMIALS

We assume that y is a function of x and can be represented with a polynomial series of x .

$$y = \sum_{k=0}^{\infty} d_k x^k \quad |x| \leq X_M$$

If we retain only the first n terms of the series

$$y_{(n)T} = \sum_{k=0}^n d_k x^k$$

we have a truncation error which is given by

$$(y - y_{(n)T}) = \sum_{k=n+1}^{\infty} d_k x^k$$

We know N measured values of y , say $\{y_i^0\}$ $i = 1, \dots, N$ corresponding to N values of x , say $\{x_i\}$ $i = 1, \dots, N$, and each measurement of y_i has a standard deviation σ .

We fit a polynomial to the data in order to: 1) evaluate the best estimates of the truncated series and 2) determine what value of n minimizes the sum of squared truncation error and standard deviation of the truncated series. In other words, we want to find those terms of the series which are significant with respect to the measured data.

The first step is to represent the truncated polynomial series by means of a set of orthogonal polynomials. A set of orthogonal polynomials over the N points $\{x_i\}$ is computed by means of the recursive relationships:

$$p_0(x) = 1$$

$$p_k(x) = (x - a_k) p_{k-1}(x) - b_k p_{k-2}(x) \quad k = 1, 2, 3, \dots$$

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$$a_k = \frac{\left[x p_{k-1}^2(x) \right]}{\left[p_{k-1}^2(x) \right]} \quad b_k = \frac{\left[p_{k-1}^2(x) \right]}{\left[p_{k-2}^2(x) \right]}$$

$$a_0 = b_0 = b_1 = 0$$

where $p_k(x)$ is a k -degree polynomial

and $[]$ means summation over the N points $\{x_i\}$

These polynomials are orthogonal, that is

$$\left[p_h p_k \right] = 0 \quad h \neq k$$

The truncated series

$$y_{(n)T} = \sum_{k=0}^n d_k x^k$$

can be represented by means of the set of orthogonal polynomials with

$$y_{(n)} = \sum_{k=0}^n c_k p_k(x)$$

where the coefficients c_k are linearly related to the coefficients d_k through the system of equations

$$c_k = \frac{1}{\left[p_k^2(x) \right]} \sum_{h=k}^n d_h \left[x^h p_k(x) \right]$$

The average of the squared truncation error is computed by means of

$$r_{(n)}^2 = \frac{1}{N} \left[\left(y - y_{(n)} \right)^2 \right] = \frac{1}{N} \sum_{k=n+1}^{\infty} c_k^2 \left[p_k^2(x) \right]$$

The best estimates \bar{c}_k of the coefficients c_k $k=1, \dots, n$ are immediately evaluated with the measured values $\{y_i^0\}$ $i=1, \dots, N$ from the relationships

$$\bar{c}_k = \frac{\left[y_i^0 p_k(x) \right]}{\left[p_k^2(x) \right]}$$

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with a standard deviation given by

$$\sigma_k^2 = \frac{\sigma^2}{p_k^2(x)}$$

where σ is the standard deviation of the measured values y_i^0 .

Employing the estimates \bar{c}_k of c_x gives estimates $\bar{y}_{(n)}$ of $y_{(n)}$

$$\bar{y}_{(n)} = \sum_{k=0}^n \bar{c}_k p_k(x)$$

with an average standard deviation given by

$$\sigma_{\bar{y}_{(n)}}^2 = \frac{n}{N} \sigma^2$$

The best value of n is therefore the one that minimizes the sum of standard deviation and truncation:

$$r_{(n)}^2 + \sigma_{\bar{y}_{(n)}}^2 = \min$$

Since

$$r_{(n)}^2 = r_{(n+1)}^2 + \frac{1}{N} \bar{c}_{n+1}^2 \left[p_{n+1}^2(x) \right]$$

$$r_{(n)}^2 = r_{(n-1)}^2 - \frac{1}{N} \bar{c}_n^2 \left[p_n^2(x) \right]$$

the minimum is obtained when

$$\bar{c}_{n+1}^2 \left[p_{n+1}^2(x) \right] < \sigma^2 < \bar{c}_n^2 \left[p_n^2(x) \right]$$

For practical purposes, an estimate of σ^2 is obtained from

$$\begin{aligned} \sigma^2 &= \frac{1}{N-n} \left[(y^0 - \bar{y}_{(n)})^2 \right] \\ &= \frac{1}{N-n} \left(\left[(y^0)^2 \right] - \sum_{k=0}^n \bar{c}_k^2 \left[p_k^2(x) \right] \right) \end{aligned}$$

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APPENDIX C

SOLUTION OF THE HOMOLOGY EQUATION

The homology equation identifies the point Q_2 on the second trace which corresponds to a given point Q_1 on the first trace. From the notation of Section 7.4, we have

$$S_2 \overline{Q_2} \cdot (S_1 \overline{S_2} \times S_1 \overline{Q_1}) = 0$$

In this appendix we show the explicit derivation of the homology equation and a method for its solution.

Consider a point Q_1 on the first trace. The vector $\overline{S_1 Q_1}$ has the coordinates

$$q_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \\ f_1 \end{pmatrix}$$

in the first camera system and has coordinates

$$S_1 Q_1 \equiv T_1 q_1$$

in the geocentric system.

Let us indicate with \overline{E} the geocentric system coordinates of the vector \overline{E} that results from the cross product of $\overline{S_1 S_2}$ and $\overline{S_1 Q_1}$

$$\overline{E} = (\overline{S_1 S_2} \times \overline{S_1 Q_1})$$

We change the coordinates of \overline{E} to the second camera system

$$e = T_2' \overline{E} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

In this same system, the second trace is represented by the relations

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$$\begin{bmatrix} q_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \\ f_2 \end{pmatrix} \\ \eta_2 = \eta_2(\xi_2) \end{bmatrix}$$

Therefore, the homology equation becomes

$$e_1 \xi_2 + e_2 \eta_2(\xi_2) + e_3 f_2 = 0$$

where ξ_2 is the unknown quantity.

This equation can be solved by means of the Newton-Raphson method if the trace is represented with an analytical function of ξ_2 , i.e., if $\eta_2 = \eta_2(\xi_2)$ is analytical.

We will use, instead, a method of searching and linear interpolation since in general the trace is represented with a tabular function of ξ_2 , i.e., we have a table $\{\xi_2, \eta_2\}$.

We use c to indicate the value of the function of ξ_2

$$c = e_1 \xi_2 + e_2 \eta_2 + e_3 f_2$$

We search the table of q_2 for two consecutive entries q_{2i} and q_{2s} such that

$$c_i = e_1 \xi_{2i} + e_2 \eta_{2i} + e_3 f_2 \leq 0$$

and

$$c_s = e_1 \xi_{2s} + e_2 \eta_{2s} + e_3 f_2 \geq 0$$

Once we have found q_{2i} and q_{2s} , we assume q_2 has the interpolated value

$$q_2 = q_{2i} - \frac{c_i}{c_s - c_i} (q_{2s} - q_{2i})$$

If the value of c does not change sign we recognize that there is no homologous point on trace-2 relative to the point Q_1 on trace-1. If the value of c is a constant equal to zero, we recognize that the homology equation vanishes and, thus, the orbital arc lies on a plane through S_1 and S_2 .

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APPENDIX D

DERIVATION OF THE COVARIANCE MATRIX OF SATELLITE POSITION

This appendix presents a derivation of the covariance matrix of satellite position based on data obtained with continuous trace camera observations. Figure 12 presents the geometry and some of the notation used.

Also

$p_1 \rightarrow$ the vector $\overline{S_1 Q_1}$

$p_2 \rightarrow$ the vector $\overline{S_2 Q_2}$

$D \rightarrow$ the distance $S_1 S_2$

$b \rightarrow$ the unit vector from S_1 toward S_2

$h \rightarrow$ the unit vector normal to b , in the plane (S_1, S_2, B)
and toward the same side of B

$n \rightarrow$ the vector $(b \times h)$

We define the angles $\gamma_1 \gamma_2 \gamma_3$ by means of the vector products

$$e_1 = (p_1 \times b) = - |p_1| \sin \gamma_1 n$$

$$e_2 = (p_2 \times b) = - |p_2| \sin \gamma_2 n$$

$$e_3 = (p_2 \times p_1) = - |p_1| |p_2| \sin \gamma_3 n$$

where $|p|$ stands for magnitude of vector p .

The unit vectors along p_1 and p_2 are given by

$$\frac{p_1}{|p_1|} = \cos \gamma_1 b + \sin \gamma_1 h$$

$$\frac{p_2}{|p_2|} = \cos \gamma_2 b + \sin \gamma_2 h$$

From Figure 12 and Section 7.4 we have the position of the satellite

$$B = S_1 + \Gamma_B \frac{p_1}{|p_1|}$$

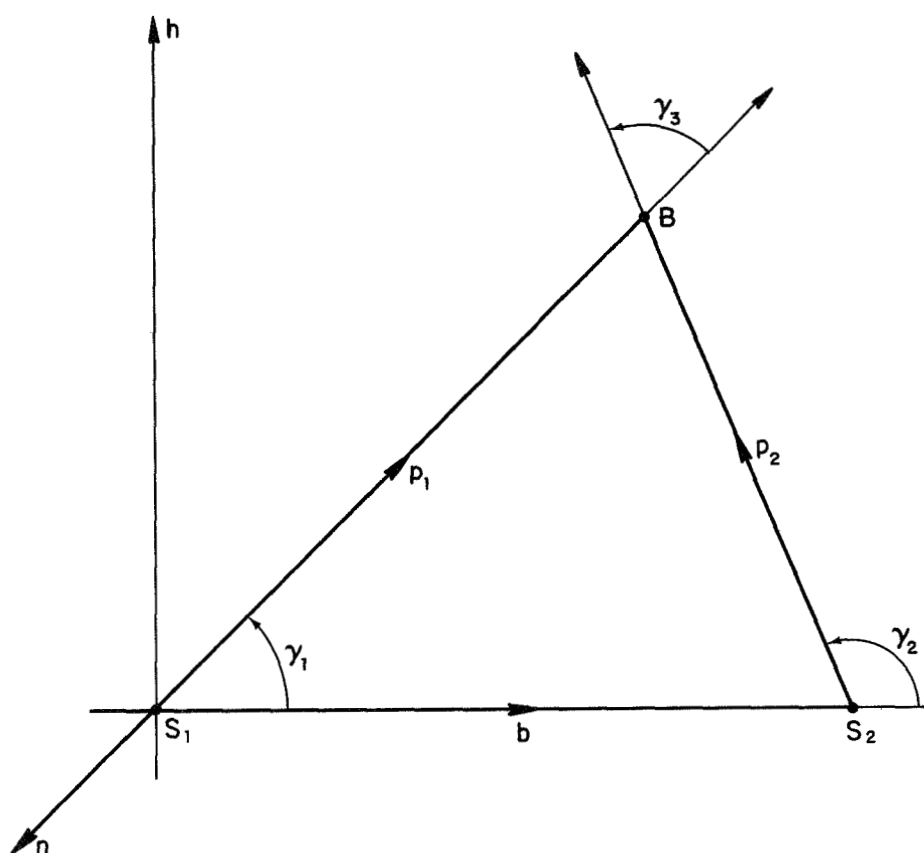


Figure 12 Geometry of Error Propagation to Satellite Position

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where

$$\Gamma_B = D \frac{\sin \gamma_2}{\sin \gamma_3}$$

The differential of B is therefore given by

$$\partial B = \Gamma_B \left(\frac{p_1}{|p_1|} \frac{\partial \Gamma_B}{\Gamma_B} + \partial \left(\frac{p_1}{|p_1|} \right) \right)$$

The second term within the parentheses can be computed from

$$\partial \left(\frac{p_1}{|p_1|} \right) = (-\sin \gamma_1 b + \cos \gamma_1 h) \partial \gamma_1$$

since b and h are not affected by measurement errors.

The first term within the parentheses can be computed from

$$\frac{\partial \Gamma_B}{\Gamma_B} = \frac{\cos \gamma_2}{\sin \gamma_2} \partial \gamma_2 - \frac{\cos \gamma_3}{\sin \gamma_3} \partial \gamma_3$$

Since

$$\gamma_3 = \gamma_2 - \gamma_1$$

the differential can be reduced to

$$\frac{\partial \Gamma_B}{\Gamma_B} = \frac{\cos \gamma_3}{\sin \gamma_3} \partial \gamma_1 - \frac{\sin \gamma_1}{\sin \gamma_2 \sin \gamma_3} \partial \gamma_2$$

Substituting this expression in the equation of ∂B and taking into account the equation of $\partial(p_1/|p_1|)$, reduces the equation of ∂B to

$$\partial B = D \frac{\sin \gamma_1 \sin \gamma_2}{\sin^2 \gamma_3} \left[\frac{p_2}{|p_2|} \frac{\partial \gamma_1}{\sin \gamma_1} - \frac{p_1}{|p_1|} \frac{\partial \gamma_2}{\sin \gamma_2} \right]$$

We note that the difference between the two indices is due to the assumed direction for b that is from S_1 to S_2 . Interchanging the indices will also invert the direction of b and therefore the sign.

To evaluate $\partial \gamma / \sin \gamma$ we note that

$$\frac{\partial \gamma}{\sin \gamma} = -\frac{1}{2} \frac{\cos \gamma}{\sin^2 \gamma} \partial (\ln \cos^2 \gamma)$$

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and

$$\cos^2 \gamma = \frac{(\mathbf{p} \cdot \mathbf{b})^2}{(\mathbf{p} \cdot \mathbf{p})}$$

Therefore

$$\begin{aligned} \partial(\ln \cos^2 \gamma) &= \partial \ln \left(\frac{(\mathbf{p} \cdot \mathbf{b})^2}{(\mathbf{p} \cdot \mathbf{p})} \right) \\ &= 2 \frac{\partial(\mathbf{p} \cdot \mathbf{b})}{(\mathbf{p} \cdot \mathbf{b})} - \frac{\partial(\mathbf{p} \cdot \mathbf{p})}{(\mathbf{p} \cdot \mathbf{p})} \\ &= 2 \left(\frac{\mathbf{b}'}{(\mathbf{p} \cdot \mathbf{b})} - \frac{\mathbf{p}'}{(\mathbf{p} \cdot \mathbf{p})} \right) \partial \mathbf{p} \end{aligned}$$

since the scalar product of two vectors is the product of the transpose of the first times the second.

Finally, we obtain

$$\frac{\partial \gamma}{\sin \gamma} = \frac{\cos \gamma}{\sin^2 \gamma} \left(\frac{\mathbf{p}'}{(\mathbf{p} \cdot \mathbf{p})} - \frac{\mathbf{b}'}{(\mathbf{p} \cdot \mathbf{b})} \right) \partial \mathbf{p}$$

Substitution in the last expression of ∂B gives

$$\begin{aligned} \partial B &= D \frac{\sin \gamma_1}{\sin \gamma_3} \frac{\sin \gamma_2}{\sin \gamma_3} \frac{p_2}{|p_2|} \frac{\cos \gamma_1}{\sin^2 \gamma_1} \left(\frac{p_1'}{(\mathbf{p}_1 \cdot \mathbf{p}_1)} - \frac{b'}{(\mathbf{p}_1 \cdot \mathbf{b})} \right) \partial p_1 \\ &\quad - D \frac{\sin \gamma_2}{\sin \gamma_3} \frac{\sin \gamma_1}{\sin \gamma_3} \frac{p_1}{|p_1|} \frac{\cos \gamma_2}{\sin^2 \gamma_2} \left(\frac{p_2'}{(\mathbf{p}_2 \cdot \mathbf{p}_2)} - \frac{b'}{(\mathbf{p}_2 \cdot \mathbf{b})} \right) \partial p_2 \end{aligned}$$

We note that

$$S_1 B = D \frac{\sin \gamma_2}{\sin \gamma_3} \frac{p_1}{|p_1|} = D \frac{(\mathbf{e}_1 \cdot \mathbf{e}_2)}{(\mathbf{e}_1 \cdot \mathbf{e}_3)} p_1$$

$$S_2 B = D \frac{\sin \gamma_1}{\sin \gamma_3} \frac{p_2}{|p_2|} = D \frac{(\mathbf{e}_1 \cdot \mathbf{e}_2)}{(\mathbf{e}_2 \cdot \mathbf{e}_3)} p_2$$

$$\frac{\sin \gamma_2}{\sin \gamma_3} \cos \gamma_1 = \frac{(\mathbf{e}_1 \cdot \mathbf{e}_2)}{(\mathbf{e}_1 \cdot \mathbf{e}_3)} (\mathbf{p}_1 \cdot \mathbf{b})$$

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$$\frac{\sin \gamma_1}{\sin \gamma_3} \cos \gamma_2 = \frac{(e_1 \cdot e_2)}{(e_2 \cdot e_3)} (p_2 \cdot b)$$

$$\sin^2 \gamma_1 = \frac{(e_1 \cdot e_1)}{(p_1 \cdot p_1)}$$

$$\sin^2 \gamma_2 = \frac{(e_2 \cdot e_2)}{(p_2 \cdot p_2)}$$

Therefore, the relationship between variation of satellite coordinates and variation of trace points is

$$\begin{aligned} \partial B = & \frac{(e_1 \cdot e_2)}{(e_1 \cdot e_3) (e_1 \cdot e_1)} S_2 B \left((p_1 \cdot b) p_1' - (p_1 \cdot p_1) b' \right) \partial p_1 \\ & - \frac{(e_1 \cdot e_2)}{(e_2 \cdot e_3) (e_2 \cdot e_2)} S_1 B \left((p_2 \cdot b) p_2' - (p_2 \cdot p_2) b' \right) \partial p_2 \end{aligned}$$

This relationship can be expressed by

$$\partial B = a (b_1 M_1 \partial p_1 - b_2 M_2 \partial p_2)$$

where $a = (e_1 \cdot e_2)$

$$b_i = \frac{1}{(e_i \cdot e_3) (e_i \cdot e_i)}$$

and

$$M_i = (p_i \cdot b) S_j B p_i' - (p_i \cdot p_i) S_j B b'$$

$$i = 1, 2 \quad j = 1, 2 \quad i \neq j$$

The covariance matrix of the satellite coordinates' random errors is

$$\begin{aligned} CV = & \langle \partial B \partial B' \rangle \\ = & a^2 \left(b_1^2 M_1 C_1 M_1' + b_2^2 M_2 C_2 M_2' \right) \end{aligned}$$

where

$$C_i = \langle \partial p_i \partial p_i' \rangle \quad i = 1, 2$$

are the covariance matrices of the random errors of the trace points'.

CSC

To evaluate the product $M_i C_i M_i'$ in practice, we observe that covariance matrices of the C_i 's are known in the camera coordinates [Section 7.5]

$$C_i = \frac{\sigma_i^2}{1 + \eta_i'^2} \begin{pmatrix} \eta_i'^2 & -\eta_i' & 0 \\ -\eta_i' & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad i = 1, 2$$

We can express them as the product of a vector d times its transposed vector d'

$$C_i = d_i d_i' \quad i = 1, 2$$

where d_i is the vector representing ∂p_i in the camera system

$$d_i = \frac{\sigma_i}{\sqrt{1 + \eta_i'^2}} \begin{pmatrix} -\eta_i' \\ 1 \\ 0 \end{pmatrix} \quad i = 1, 2$$

Therefore, we operate directly on the vector d before performing the products $M C M'$. We compute the vector

$$D_i = M_i T_i \begin{pmatrix} -\eta_i' \\ 1 \\ 0 \end{pmatrix} \quad i = 1, 2$$

where T_i transforms the coordinates from camera system to geocentric system. Doing so, we obtain

$$M_i C_i M_i' = \frac{\sigma_i^2}{1 + \eta_i'^2} D_i D_i' \quad i = 1, 2$$

Finally, the covariance matrix CV is given by

$$CV = a^2 (k_1^2 D_1 D_1' + k_2^2 D_2 D_2')$$

where

$$k_i = \frac{\sigma_i}{(e_i \cdot e_3) (e_i \cdot e_i) (1 + \eta_i'^2)^{1/2}} \quad i = 1, 2$$

This method reduces the direct product of square matrices to axes transformations and product of vectors. Thus, we can obtain a considerable saving of computer time.

CSC

The values of σ_1 and η_1^i are evaluated in practice by means of fitting a polynomial through the trace with the procedure described in Appendix B.

CSC

APPENDIX E

COMPUTATION OF MATRICES IN THE LEAST SQUARE SOLUTION

The matrices involved in the normal equation of the least square procedure are evaluated from the partial derivatives of the constraints equations. These equations are given by [Section 7.6]

$$G(S, P, p) = f(y - y_s) - \eta(z - z_s)$$

$$\eta = \eta(\xi)$$

$$\xi = f \frac{x - x_s}{z - z_s}$$

To evaluate the partial derivatives we use the fact that η is a function of ξ , and ξ is a function of $(x - x_s)$ and $(z - z_s)$.

We indicate the derivative of the trace equation by

$$\eta' = \frac{d\eta}{d\xi}$$

Therefore, we have

$$G_P = \begin{pmatrix} -\eta' (z - z_s) \frac{\partial \xi}{\partial x} \\ f - \eta' (z - z_s) \frac{\partial \xi}{\partial y} \\ -\eta' (z - z_s) \frac{\partial \xi}{\partial z} - \eta \end{pmatrix}$$

$$G_P = \begin{pmatrix} -\eta' (z - z_s) \\ - (z - z_s) \\ 0 \end{pmatrix}$$

$$G_S = - G_P$$

CSC

To complete the evaluation

$$\frac{\partial \xi}{\partial x} = \frac{f}{z - z_s}; \frac{\partial \xi}{\partial y} = 0; \frac{\partial \xi}{\partial z} = -\frac{\xi}{z - z_s}$$

Substitution in the expression of G_P gives

$$G_P = \begin{pmatrix} -f\eta' \\ f \\ -\eta + \xi\eta' \end{pmatrix}$$

$$G_P = -(z - z_s) \begin{pmatrix} \eta' \\ 1 \\ 0 \end{pmatrix}$$

$$G_S = \begin{pmatrix} f\eta' \\ -f \\ \eta - \xi\eta' \end{pmatrix}$$

The value of the coefficients C_λ [Section 7.6] can be directly computed from

$$C_\lambda = \overline{G}_P' C V \overline{G}_P + G_P' C G_P$$

CV is given in the geocentric system; therefore, we have to convert G_P from camera system to geocentric system

$$\overline{G}_P = T G_P$$

Further, from Section 7.5

$$C = \frac{\sigma^2}{1 + \eta'^2} \begin{pmatrix} \eta'^2 & -\eta' & 0 \\ -\eta' & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where σ^2 is the standard deviation of the plate measurements. Thus,

$$G_P' C G_P = \sigma^2 (z - z_s)^2 \frac{(1 - \eta'^2)^2}{(1 + \eta'^2)}$$

CSC

Once the values of C_λ are known, the normal matrix is

$$N_S = \sum_{i=1}^N \frac{1}{C_{\lambda i}} G_S G_S' + W_S$$

where

$$G_S G_S' = \begin{pmatrix} f^2 \eta'^2 & -f^2 \eta' & f \eta' (\eta - \xi \eta') \\ -f^2 \eta' & f^2 & -f (\eta - \eta' \xi) \\ f \eta' (\eta - \eta' \xi) & -f (\eta - \eta' \xi) & (\eta - \xi \eta')^2 \end{pmatrix}$$

and the known vector is

$$E_S = \sum_{i=1}^N \frac{G_o}{C_{\lambda i}} G_S$$